

The pure theory of large two-candidate elections

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INTRODUCTION

A long-standing goal of political theorists has been the development of a coherent, consistent, and nonvacuous theory of elections, particularly of those using majority rule. If possible, such a theory is to be based on rational individual and group behavior. In spite of extensive effort, recent writings (see, for example, Ordeshook and Shepsle, 1982) reveal that many may now be prepared to give up this research program on the grounds that no such model exists. There appear to be two main stumbling blocks to a consistent theory based on the rational behavior of participants: (1) the theoretical proposition that, given any realistic assumption about the cost of voting, rational voters will not participate in elections, and (2) even if they vote, majority-rule equilibria rarely exist. The first result is obviously contradicted by

*Northwestern University. The research for this paper has been supported by NSF grant SES8106896 which is gratefully acknowledged. The paper itself was prepared for presentation at the Conference on Political Economy at Carnegie-Mellon University in June 1983. Without that incentive it might still be unwritten. I have had the help and advice of many people--some of whom disagree with the conclusions. Among these were the participants in workshops at Cal Tech, Iowa, Tulane, Indiana, Stanford, and Northwestern University. The exacting standards of Howard Rosenthal are responsible for much of whatever quality in presentation exists. I thank him for this help and advice. Peter Coughlin found early lapses. An early version was presented at the Public Choice Society meetings, as my presidential address, in 1981.

the facts; the second means that the theory as we know it is fundamentally flawed. Faced with these results, those who have not given up on political theory altogether have gone in two other obvious directions. They have either given up on "rational" behavior (see, for example, Hinich et al., 1972; Coughlin, 1979), or they have given up on "equilibrium" models and have turned to "process" models (for example, Kramer, 1977.)

It is my belief that this retreat is premature. In particular, I intend to show in this paper that even under assumptions of extremely rational behavior, it is possible to combine voters, who may or may not vote depending on the benefits and costs, with candidates who game against one another and end with equilibria which not only exist but

Figure 1

	RATIONAL CHOICE	PROBABILISTIC VOTING
NO ABSTENTIONS	TRADITIONAL	COUGHLIN
ABSTENTIONS	THIS PAPER	HINICH, LEDYARD & ORDESHOOK

also have a remarkable social-welfare property. The approach is a straightforward extension of the now standard spatial competition model of elections. Voters have preferences (utility functions) over issues, candidates choose a platform (a point in the issue space), and then voters vote for their most preferred candidate--in two-candidate elections--if and only if the expected benefits from so doing outweigh the costs. Given this voter behavior, candidates are assumed to maximize expected plurality (a very good approximation of the probability of winning). A full general equilibrium occurs when no voter or candidate wishes to alter his strategy.

To show where the theory posed in this paper fits into the literature on the theory of majority-rule elections, I refer the reader to Figure 1 in which existing theories are divided into four "boxes" depending upon the assumptions concerning voting behavior. In the traditional theory it is assumed that all vote (no abstentions) and that choice behavior is rational (some form of utility maximization). It is this theory for which equilibria rarely exist. Hinich et al. changed both of these behavioral hypotheses by allowing abstentions due to indifference, alienation, etc. and by modeling the decision to vote as probabilistic while leaving the choice of candidate based on utility. Although equilibria exist in this modification, voting behavior is somewhat ad hoc and certainly not rational-choice-based. Coughlin maintained the traditional assumption of no abstentions but removed the voter's choice of candidate from rational theory. Instead, he adopted the decision-theoretic-framework of Luce (1959, 1977) by assuming that choice is probabilistic, where probabilities are proportional to utility. With this model of voter behavior, equilibria exist and have no interesting welfare property albeit different from that in this paper. It is not known what occurs in Coughlin's models if abstentions are allowed.¹

The model in this paper departs from the traditional by allowing

¹Stop Press: I have recently seen Coughlin (1983b) in which abstentions are allowed but only on an aggregate basis. Individual behavior remains unspecified.

rational abstention behavior.² We begin by recognizing the obvious fact that when voters make the decision to vote, they do not know how many others have voted, or plan to vote, or, especially, how the others have voted. They face a decision -- or better, a game -- under uncertainty similar in spirit to a sealed-bid auction. In modeling this simultaneous decision problem for all voters we impose as much rationality as possible -- rational choice and rational expectations -- and arrive at a model in which turnout is neither the 100% nor the 0% that has traditionally been implied by rational-choice models. It is this model of the voters' behavior which constitutes the "new" component of the theory in this paper. Most of the rest of our model is standard, although the implications derived from this combination of new and old are not.

In Section I we describe the behavior of a single voter in much the same way as that posed by Downs (1957), Tullock (1967), and others. In Section II we consider the simultaneous behavior of all voters and present the equilibrium concept first introduced in Ledyard (1981). In Section III we define and describe both the behavior of candidates and the equilibrium which arises when all actors -- candidates and voters -- are combined into a general equilibrium. In Section IV we explore the welfare properties of those equilibria, in Section V we examine the existence of equilibrium, and concluding remarks are added in Section VI.

I. THE VOTER

The voter is assumed to choose whether to vote or abstain, as well as for whom to vote, consistent with the expected utility hypothesis. This model has already received much attention in the literature so I will not dwell on its rationale but will immediately turn to the notation and definitions. The interested reader can consult Ferejohn and Fiorina (1974) for a good survey.

² I am finally answering the complaints of Slutsky (1975) about the ad hoc and unrealistic nature of our earlier paper (Hinich et al., 1972). In spite of my efforts he remains unconvinced of the "reality" of the model.

We assume now that there are only two candidates, A and B. Candidate A chooses a platform which we denote by A and candidate B chooses a platform denoted by B . We assume that the voter knows the candidates' choices and has a utility function over all possible platforms, R , given by $u(R, x)$ where x represents the appropriate utility parameters for this voter. We assume throughout that u is continuous in R . We sometimes call x the "type" of this voter. If this voter decides to go to the polls, he will cast his vote for A over B if and only if $u(A, x) > u(B, x)$. We assume the voter receives no consumption benefit from voting. Therefore, whether this voter will vote instead of abstaining depends on a simple benefit-cost calculation. The expected benefits from voting are equal to the probability of affecting the outcome times the gain from so doing. Letting P be the probability that this particular voter will alter the outcome, and assuming that $u(A, x) > u(B, x)$, the expected benefits are $(P)(u(A, x) - u(B, x))/2$. The utility difference is divided by 2, since a voter affects the outcome only if he creates a tie or breaks one. Assuming that ties are broken by a fair coin toss, the gain from either event is the utility difference divided by 2. We assume that the voter faces a known cost of voting equal to $c > 0$ and that this cost enters the utility calculation linearly. Therefore, if candidate A wins and the voter had gone to the poll, he receives $u(A, x) - c$ in utility.

In order to complete this model of rational-voting behavior, we must provide a basis for the voter's beliefs about P_a and P_b , where P_j is the probability that candidate j either ties the other or loses by one vote. We assume, at this point, that the voter knows the probability that a voter, randomly selected from all other voters, will vote for A, vote for B, or abstain. (We will see in the next section how these can be estimated.) Using these probabilities, denoted, respectively, Q_a , Q_b , and Q_0 , where $Q_a + Q_b + Q_0 = 1$, it is a standard exercise to calculate the probability of a tie when there are n other voters. It is also easy to calculate the probability that A loses to B by one vote. Adding these we find that $P_a = f(Q_a, Q_b)$ where $f(z, y) =$

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!k!(n-2k)!} z^k y^k (1-z-y)^{n-2k} + \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{(k+1)!k!(n-2k-1)!} z^{k+1} y^k (1-z-y)^{n-2k-1}$$

A symmetric calculation yields $P_b = f(Q_b, Q_a)$. Gathering this all together we have described the voter.

A voter with characteristics (x, c) who is faced with a choice between two candidates, A and B, and who thinks the probability that a randomly selected voter will vote for candidate j is Q_j will

- (a) vote for A if $c < (P_a/2)(u(A, x) - u(B, x))$,
- (b) vote for B if $c < (P_b/2)(u(B, x) - u(A, x))$,
- (c) abstain otherwise,

where $P_a = f(Q_a, Q_b)$, $P_b = f(Q_b, Q_a)$ and f is defined above.

This model assumes rational behavior in the form of expected utility maximization, no income effects, no candidate specific preferences other than the platform choice, positive costs of voting, and knowledge by the voter of x, c, A, B, P_a , and P_b .

At this point most writers reach an unsettling conclusion. "Since the expected benefit from voting is obviously small (if $Q_a = Q_b$ and $Q_0 = 0$ then P_a and P_b are of order of magnitude $1/2n$ -- see Chamberlain and Rothschild, 1981), and since the cost of voting is not small, no rational voter will ever vote in large elections. Therefore, something must be wrong with the theory." This is not an unreasonable conclusion but the analysis is incomplete, since it is based on a partial equilibrium view which is simply not appropriate. If this voter and others are embedded in a general equilibrium model, the apparent failure of rational choice to explain voting disappears. We turn to that task next.

II. RATIONAL VOTERS' EQUILIBRIUM

We now explore what happens when voters take into account the fact that other voters are also rational. The logic is simple and compelling and is contained in Ferejohn and Fiorina (1974). If everyone is rational and carries out the partial equilibrium calculus in the previous section then, presumably, no one will vote. But then the probability of a tie is 1. If this is true and if these same rational, partial equilibrium nonvoters redo their calculus, most will find that it is now definitely in their interest to vote; they will be able to determine the outcome by

themselves. And so on. Somewhere between no one voting and everyone voting lies a situation in which some vote and in which the probability of a tie is consistent with those numbers and with the beliefs of all voters. It is this stable, rational, intermediate situation that we capture in the voters' equilibrium defined below.

To close the partial equilibrium model in the previous section, it remains only to specify how a voter estimates Q_a and Q_b . We assume that it is common knowledge among all voters that each voter is rational and, therefore, that each follows the model of Section I. What is not known to each voter, and never will be, are the values of the others' characteristics (x, c) . We do, however, assume that the distribution of these characteristics is known to all by the density functions $h(c)$ and $g(x)$. That is, c and x are independently distributed, where $g(x)$ is the probability that a randomly selected voter will have characteristic x , and $h(c)$ is the probability that a randomly selected voter will have a cost of voting equal to c .³

Given these densities, one can compute the probability that a randomly selected voter will vote for a candidate. We already know that the voter will vote for A if and only if her characteristic, (x, c) , satisfies

$$c < (P_a/2)(u(A, x) - u(B, x)).$$

Using the densities g and h we can compute that the probability of this is⁴

$$Q_a = \int_{X+(A,B)} H((P_a/2)(u(A,x) - u(B,x))) g(x) dx$$

where $X+(A,B) = \{x | u(A,x) > u(B,x)\}$ and $H(r) = \int_0^r h(c) dc$. Writing this

³The assumption of independence is made only for expositional convenience. The eager reader can easily show that correlation between c and x in a density function like $g(x, c)$ can be accommodated without destroying the results detailed below.

⁴Note that if $A = B$, then $Q_a = 0$ since $u(A, x) = u(B, x)$, and $H(0) = 0$.

as $Q_a = t(P_a, A, B; g, h)$, it is easy to show that $Q_b = t(P_b, B, A; g, h)$ and $Q_0 = 1 - Q_a - Q_b$.

We can thus compute Q_a and Q_b from P_a and P_b . In the previous section we computed P_a and P_b from Q_a and Q_b . A fully rational voter with fully rational expectations will require these calculations to be consistent with one another and will be able to compute the values of Q and P for which consistency obtains.

Given the densities on characteristics, h and g , and given the candidate platforms A and B , we call $\langle P_a, P_b, Q_a, Q_b \rangle$ a *RATIONAL VOTERS' EQUILIBRIUM* if and only if

$$P_a = f(Q_a, Q_b) \text{ and } P_b = f(Q_b, Q_a) \text{ and}$$

$$Q_a = t(P_a, A, B; g, h) \text{ and } Q_b = t(P_b, B, A; g, h),$$

where $f(\cdot)$ is defined in Section I and $t(r, s, w; g, h)$ is defined above.

As an aside the reader should note that if we were to model the voters as playing a game of incomplete information, as is done in modeling auctions, the three pure strategies would be vote A , vote B , and abstain, and the Bayes equilibria of that game would be exactly the Rational Voters' Equilibrium defined above. I chose the approach above for its expositional simplicity.

To complete this section, we consider several properties of the rational voters' equilibrium.

PROPOSITION 1: (EXISTENCE). If $H(c) \in C$ (that is, if H is continuous), then a rational voters' equilibrium exists.

PROOF: If $A = B$, then $Q_a = Q_b = 0$, $P_a = P_b = 1$ is an equilibrium. If $A \neq B$, then define the functions $P_a = N1(P_a, P_b) = f(t(P_a, A, B), t(P_b, B, A))$ and $P_b = N2(P_a, P_b) = f(t(P_b, B, A), t(P_a, A, B))$. It is easy to show that $N1$ and $N2$ are continuous in (P_a, P_b) since f is polynomial and therefore continuous, while t is continuous in P since H is by assumption. Further, $N1$ and $N2$ map $[0,1] \times [0,1]$ into itself. Therefore, Brouwer's fixed-point theorem can be applied. There is at least one pair $P^* = (P_a^*, P_b^*)$ such that $P^* = N(P^*)$. Let $Q_a^* = t(P_a^*, A, B)$ and $Q_b^* = t(P_b^*, B, A)$. Then (P^*, Q^*) is a rational voters'

equilibrium.

Q.E.D.

PROPOSITION 2: (SYMMETRY). (P_a, P_b, Q_a, Q_b) is a rational voters' equilibrium given (A, B) if and only if (P_b, P_a, Q_b, Q_a) is a rational voters' equilibrium given (B, A) .

PROOF: Immediate.

Q.E.D.

This is the first of several propositions concerning the symmetry of the model in this paper. The primary reason for symmetry is that we have assumed that voters care about the platform which candidates adopt and not the name of the candidate.

The next property is of interest for its implications about the voting probabilities in equilibrium.

PROPOSITION 3: In any rational voters' equilibrium, $(P_a - P_b) = (Q_b - Q_a)^F$ where $F \geq 0$.

PROOF: $P_a - P_b = f(Q_a, Q_b) - f(Q_b, Q_a) =$

$$(Q_b - Q_a) \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{k!k+1!(n-2k-1)!} x^k y^{n-2k-1}.$$

Q.E.D.

It should be noted for completeness that $F = 0$ if and only if the number of voters is even and $Q_a + Q_b = 1$, (i.e., turnout is 100%).

Another interesting property of equilibrium is uniqueness, or lack thereof. We have two propositions to present, both of which depend on the turnout probabilities.

Definition: (Maximum Turnout Probability). Given the candidates platforms, A and B , and given the distribution of voters' characteristics, we can compute an upper limit on turnout which is independent of the particular voter equilibrium arrived at. In particular, let

$$M(A, B, g, h) = \int H((1/2)(u(A, x) - u(B, x)))g(x)dx.$$

We call $M(\cdot)$ the maximum turnout probability.

We have defined $M(\cdot)$ this way since M is the probability that a randomly selected voter will go to the polls if he thinks the probability of a tie is 1. To see this, remember that

$$Q_a + Q_b = \int_{X+(A,B)} H((P_a/2)(u(A,x) - u(B,x)))g(x)dx \\ + \int_{X+(B,A)} H((P_b/2)(u(B,x) - u(A,x)))g(x)dx.$$

Let $P_a = P_b = 1$. The observation follows immediately, since H is a distribution function and $H' \geq 0$.

$M(\cdot) = 0$ if no rational voter will go to the polls even when the probability of influencing the election is 1. For an example, assume $H(0) = 0$ and let $A = B$.

PROPOSITION 4: (UNIQUENESS 1). If $M(A,B,g,h) = 0$ then $(1,1,0,0)$ is the unique rational voters' equilibrium.

PROOF: Under the hypothesis, $Q_a = Q_b = 0$ for all values of P_a and P_b since $H(c) \geq H(c')$ whenever $c > c'$. But if Q_a and Q_b are 0 it follows that $P_a = P_b = 1$.

Q.E.D.

It would be helpful if we were also able to exhibit a proposition listing sufficient conditions for the uniqueness of the voter equilibrium when the maximum turnout probability is positive. Unfortunately, I have not yet discovered such a result. It is true, however, that if the candidates' platforms are close enough, then M is near 0 and the equilibrium will be both unique and continuous in (A,B) .

PROPOSITION 5: (UNIQUENESS 2). Suppose $M(A,B,g,h) > 0$, $u(\cdot, x) \in C^1$ for all x , and $H(c) \in C^1$ for all c . If $M(A,B,g,h)$ is near 0, (which is true, for example, if A is near B), the equilibrium (P_a, P_b, Q_a, Q_b) is unique

and is C^1 in A and B (for $A \neq B$).⁵

PROOF: Let $Q_a(P_a) = t(P_a, A, B)$ and $Q_b(P_b) = t(P_b, B, A)$. (P, Q) is an equilibrium if and only if P solves

$$P_a - f(Q_a(P_a), Q_b(P_b)) = 0 \text{ and}$$

$P_b - f(Q_b(P_b), Q_a(P_a)) = 0$. The Jacobian of this system of equations is

$$s = \begin{vmatrix} 1 - f_1(Q_a, Q_b)Q_a' & -f_2(Q_a, Q_b)Q_b' \\ -f_2(Q_b, Q_a)Q_a' & 1 - f_1(Q_b, Q_a)Q_b' \end{vmatrix}$$

$$Q_a' = \partial Q_a / \partial P_a = \partial \int_{X^+} h[(P_a/2)(u(A,x) - u(B,x))] g(x) dx / \partial P_a$$

$$= \int_{X^+} h[(P_a/2)(u(A,x) - u(B,x))] g(x) \{u(A,x) - u(B,x)\} \left(\frac{1}{2}\right) dx$$

Since this integral is taken over X^+ , its value is positive. Similarly for Q_b' . From Ledyard (1981) we know that

$$f_1(x,y) = (y-x) \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{k+1!k-1!n-2k-1!} x^{k-1} y^k (1-x-y)^{n-2k-1}$$

$$- \sum_{k=1}^{\lfloor \frac{n-1}{2} \rfloor} \frac{n!}{k!k+1!n-2k!} x^k y^k (1-x-y)^{n-2k-1}$$

$$- n(1-x-y)^{n-1}$$

⁵ Q_a and Q_b may have discontinuous derivatives at $A = B$. I thank Peter Coughlin for noting this in an earlier version.

and

$$f_2(x,y) = (x-y) \sum_{k=1}^{\lfloor \frac{n}{2} \rfloor} \frac{n!}{k!(k-1)!(n-2k)!} x^{k-1} y^{k-1} (1-x-y)^{n-2k}.$$

From these it can be seen that the signs of f_i are:

	$f_1(Q_a, Q_b)$	$f_2(Q_a, Q_b)$	$f_1(Q_b, Q_a)$	$f_2(Q_b, Q_a)$
if $Q_a > Q_b$	-	+	?	-
if $Q_a = Q_b$	-	0	-	0
if $Q_a < Q_b$?	-	-	+

Since f_1, f_2 are continuous if Q_a is near Q_b then $f_1(Q_a, Q_b) < 0$ and $f_1(Q_b, Q_a) < 0$. Therefore,

$$\text{for } Q_a > Q_b \quad J = \begin{vmatrix} + & - \\ + & + \end{vmatrix}$$

$$\text{for } Q_a = Q_b \quad J = \begin{vmatrix} + & 0 \\ 0 & + \end{vmatrix}$$

$$\text{for } Q_a < Q_b \quad J = \begin{vmatrix} + & + \\ - & + \end{vmatrix}$$

Now we know that any solution must satisfy $0 \leq Q_a = Q_b \leq M(A,B)$, and $Q_a, Q_b \geq 0$. Therefore, $\|Q_a - Q_b\| \leq M(A,B)$ and if $M(A,B)$ is small enough, Q_a is always near Q_b .

J is positive definite for all such Q_a, Q_b . Therefore, the equilibrium is unique. (Gale-Nikaido, 1965)

Continuity follows from the Implicit Function Theorem

Q.E.D.

To summarize, if the maximum turnout probability is small enough (or if the candidates' platforms are close enough), the voter equilibrium is unique and C^1 in the platforms. I do not know how close is "enough." If A is not near B , it seems that multiple equilibria may be

possible.

A final comment seems in order about the amount of turnout predicted by this model. We have seen that if the maximum turnout probability is 0 or if $A = B$, then turnout is predicted to be 0. Since we have adopted a rational behavior hypothesis, one might suspect that, in fact, turnout is never positive. Such a suspicion would be false.

PROPOSITION 6: (POSITIVE EXPECTED TURNOUT). If the maximum turnout probability is positive, given A and B , then expected turnout is positive in any rational voters' equilibrium.

PROOF: Remember that expected turnout is $(n+1)(Q_a + Q_b)$. Suppose that $Q_a + Q_b = 0$. Then $Q_a = Q_b = 0$. But $f(0,0) = 1$. Therefore, $P_a = P_b = 1$. It follows that expected turnout is then $(n+1)M(A,B) > 0$ which is a contradiction.

Q.E.D.

Corollary: If there is a set of x , with positive measure, such that $u(A,x) - u(B,x) \neq 0$ and if $H(c) > 0$ when $c > 0$ (i.e., $h(c) > 0$ for all $c \geq 0$), then expected turnout will be positive in equilibrium. $M(\cdot)$ gives an upper-bound on expected turnout.

Thus, contrary to naive expectations based on partial equilibrium analysis, a rational general equilibrium consideration of voting behavior yields positive turnout in equilibrium unless each voter refuses to vote even when he knows he is the only voter.

III. THE CANDIDATE AND THE ELECTION EQUILIBRIUM

In this section, we model how candidates determine their platforms and, therefore, the outcome of the election. We begin by considering what motivates the candidates. Since this is a static model and since we have been assuming that platforms will be implemented and that the extent of implementation does not depend on the margin of victory, it seems reasonable to assume that these candidates care, ex post, only about winning. The appropriate outcome space then is simply the two-point set $\{W, L\} = \{\text{win, lose}\}$. The simultaneous choice of platforms by the candidates determines a probability distribution (R_a, R_b) on this set and the rational, expected utility-maximizing candidate A will choose

the platform to maximize $R_a * V(W) + R_b * V(L)$. Thus, this candidate will always choose to maximize the probability of winning. We assume that these candidates know the model of the previous section, or at least act as if they know it. From that model they can determine an election-outcome function, or, more properly, an outcome correspondence, which maps pairs of platforms (A,B) into sets of 4-tuples (P_a, P_b, Q_a, Q_b) . It is possible for candidates to compute various implications of their choices such as the probability of winning.

The probability A wins

Given A,B, $h(c)$, and $g(x)$, and a rational voters' equilibrium of a two-candidate election, the probability that A wins is:

$$R_a = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{r=1}^{n-2k+1} \frac{n!}{k+r! k! n-2k-r+1!} (Q_A)^{k+r} (Q_B)^k (1-Q_A-Q_B)^{n-2k-r+1}$$

$$+ \frac{1}{2} \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \frac{n!}{k! k! n-2k+1} (Q_A)^k (Q_B)^k (1-Q_A-Q_B)^{n-2k+1}$$

where $Q_a(A,B)$ and $Q_b(A,B)$ are the appropriate parts of a rational voter equilibrium for A,B.

Although the analysis can be carried out using it, this is a remarkably unwieldy function. To simplify, let us use an approximation of R_a which is appropriate for large elections. (See Hinich, 1977.) If n is large, then $Q_a - Q_b$ is a good approximation for a candidate to use in place of R_a . To see this, let $S_i = 1$ if voter i votes for A, $S_i = 0$ if i abstains, and $S_i = -1$ if i votes for B. Then A wins if and only if

$$\sum_{i=1}^{n+1} S_i > 0. \quad \text{This is true if and only if}$$

$$(1/(n+1)) \sum_{i=1}^{n+1} S_i = \bar{S} > 0. \quad \text{Since the } S_i \text{ are independently and}$$

identically distributed, it follows from a Law of Large Numbers that

$$\lim_{n \rightarrow \infty} \Pr(\bar{S} > 0) = \begin{cases} 1 & \text{if } Q_a > Q_b \\ 1/2 & \text{if } Q_a = Q_b \\ 0 & \text{if } Q_a < Q_b \end{cases}$$

Therefore, maximizing $Q_a - Q_b$ maximizes (in the limit) the probability that A wins. Using this approximation, we posit the following model of the candidates.

An alternative justification for the use of expected plurality, $Q_a - Q_b$, in place of R_a is that the equilibrium described below is the same in both cases because there are two candidates and symmetry. Basically, we can show that $R_a \geq 1/2$ if and only if $Q_a \geq Q_b$. Since equilibria will occur only at $R_a = 1/2$ or $Q_a = Q_b$, the two objectives produce the same equilibrium. We adopt $Q_a - Q_b$ for simplicity.

The candidates' objectives

In a large, two-candidate election, each candidate will try to maximize expected plurality. In particular, the objective function of candidate A is

$$W(A,B) = Q_a(A,B) - Q_b(A,B)$$

and that of candidate B is

$$V(A,B) = Q_b(A,B) - Q_a(A,B)$$

where (P_a, P_b, Q_a, Q_b) is a Rational Voters Equilibrium for the platform choices A,B.⁶ The observant reader will have already noticed a potential difficulty with this model--a rational voters equilibrium may not be unique and, therefore, the mapping $W(A,B)$ may not be a

⁶The Rational Voters' Equilibrium is defined in Section II.

function. We do have to confront this problem, but if $W(\cdot)$ and $V(\cdot)$ are unique, the above objective functions would point instantly to the appropriate behavior for the candidates in their choice of a platform, since this is a two-person zero-sum game for which game theorists are in agreement about a solution concept. We adopt the consensus solution concept with modifications because of the nonuniqueness.

The candidates' behavior

In a two-candidate large election, candidates will choose platforms (A^*, B^*) which satisfy

$$\min W(A^*, B^*) \geq \max W(A, B^*) \quad \text{for all } A$$

$$\min V(A^*, B^*) \geq \max V(A^*, B) \quad \text{for all } B$$

where $W(A, B) = [wIQ_a - Q_b = w \text{ for some rational voters' equilibrium}]$, and $V(A, B) = [vIQ_b - Q_a = v \text{ for some rational voters' equilibrium}]$. We call (A^*, B^*) a (STRONG) RATIONAL ELECTION EQUILIBRIUM.

If $W(\cdot)$ and $V(\cdot)$ are single-valued, this definition corresponds to the noncooperative equilibrium (or maximin solution) of this game. Due to the modification, we have called this a strong equilibrium since, if the candidates choose these strategies, then even if candidate A could choose from the multiple set $W(\cdot)$ after changing her strategy, she could do no better than now. Weaker equilibria may also exist since a risk-averse candidate might choose to play a strategy A^* , even though $\max W(A, B^*) > \min W(A^*, B^*)$ for another A, in order to avoid a possible loss if $\min W(A, B^*) < \max W(A^*, B^*)$. I have chosen the stronger version, since a more strategic candidate would notice that even if such a loss occurred he could regain at least a payoff of 0 by choosing $A = B^*$. Thus, no outcome which yields less than 0 to some candidate should survive as an equilibrium. A strong equilibrium has the property that each player receives 0.

PROPOSITION 7: (VALUE). If (A^*, B^*) is a strong rational election equilibrium then $W(A^*, B^*) = V(A^*, B^*) = 0$

PROOF: Given any B, since candidate A can always choose the same

platform, B, it must be true that $\min W(A^*, B^*) \geq 0$. Also, $\min V(A^*, B^*) \geq 0$. But it is easy to see that if $w \in W(A^*, B^*)$ then $-w \in V(A^*, B^*)$. Therefore, $\min W(A^*, B^*) \geq \max W(A^*, B^*) = 0$.

Q.E.D.

This means that (A^*, B^*) is a strong equilibrium if and only if $\max W(A, B^*) \leq 0$ for all A and $\max V(A^*, B) \leq 0$ for all B. Even if there are "weaker" equilibria, one suspects that only strong equilibria are permanent. We, therefore, concentrate on them.

IV. EQUILIBRIUM AND OPTIMALITY

In this section we show that if utility functions are concave and have continuous derivatives in A, and costs are distributed from zero, then all equilibria can be characterized in a remarkably simple manner; the candidates choose the same platform, the chosen platform maximizes $\int u(A, x)g(x)dx$, and no one votes. Thus, if an equilibrium exists there is a simple maximization problem by which it can be computed. We give several examples at the end of this section.

To show these properties of equilibrium, we need to establish intermediate results. The first of these occurs because of the symmetry of the model; candidates are essentially anonymous in all respects except their platforms.

PROPOSITION 8: (SYMMETRY). If (A, B) is a strong rational election equilibrium then so are (A, A) , (B, A) , and (B, B) .

PROOF: Since (A, B) is an equilibrium, $W(A, B) = 0 = V(A, B)$ from proposition 4. For all D and w if $w \in W(D, B)$ then $w \leq 0$. For all D and v if $v \in V(A, D)$ then $v \leq 0$. Further, we know that $w \in W(A, B)$ if and only if $-w \in V(A, B)$ if and only if $w \in V(B, A)$. Now suppose that $w \in V(B, D)$ for some D. Then, $-w \in W(D, B)$ which implies that $w \in W(D, B)$ and therefore $w \leq 0$. Thus, (B, B) is a strong rational election equilibrium. The rest follows in a similar manner.

Q.E.D.

Now we take up two lemmas which allow us to use calculus in the analysis of equilibrium.

LEMMA 1: (A^*, B^*) is a strong equilibrium if and only if $\int H((P_a/2)|D|)I(D)g(x)dx \leq 0$ for all A where $D = u(A, x) - u(B^*, x)$ and where $I(D) = 1$ if $D > 0$, $I(D) = 0$ if $D = 0$, and $I(D) = -1$ if $D < 0$.

PROOF: By definition (A^*, B^*) is a strong equilibrium if and only if

$$\int_{X^+} H((P_a/2)|D|)g(x)dx - \int_{X^-} H((P_b/2)|D|)g(x)dx \leq 0 \quad (1)$$

for all A. This is true if and only if

$$\int H((P_a/2)|D|)I(D)g(x)dx + \int_{X^-} [H((P_a/2)|D|) - H((P_b/2)|D|)] g(x)DX \leq 0. \quad (2)$$

for all A. This in turn is true if and only if

$$\int H((P_a/2)|D|)I(D)g(x)dx \leq 0 \quad (3)$$

for all A.

Statement (1) follows from the remark after Proposition 7 above. Statement (2) follows by adding and subtracting $\int_{X^-} H((P_a/2)|D|)g(x)dx$ to and from the left side of (1). To establish (3) takes more work. I will prove that (2) implies (3) and leave the converse to the reader. Assume that $\int H((P_a/2)|D|)g(x)dx > 0$ and that (2) is correct for some A. It must then be true that

$$\int [H((P_a/2)|D|) - H((P_b/2)|D|)] g(x)dx < 0.$$

Therefore, $P_a < P_b$. Referring to lemma 3 in Section II we see that $Q_b < Q_a$. But this implies that (1) is > 0 since (1) is $Q_a - Q_b$. This in turn implies that (2) is > 0 which contradicts our initial assumption.

Q.E.D.

LEMMA 2: Given (A^*, B^*, P_a) where P_a is a voters' equilibrium for

A^*, B^* . If A^* is "near" B^* and if $u \in C^1$ and $H \in C^1$ and if their derivatives are bounded, then

$$\frac{d \int H((P_a/2)|D|I(D))g(x)dx}{dA} = \int h((P_a/2)|D|) [(dP_a/dA)(D/2) + (P_a/2)(dD/dA)] g(x)dx.$$

PROOF: For any A and x such that $I(D) \neq 0$, we find that

$$\frac{d(H((P_a/2)|D|I(D)))}{dA} = h((P_a/2)|D|)[D/2](dP_a/dA) + (P_a/2)(dD/dA).$$

It follows that equality is also true if $I(D) = 0$.⁷ From Proposition 5 of the previous section dH/dA exists for all x, since A^* is near B^* . The Lemma then follows from the Lebesgue Dominated Convergence Theorem.

Q.E.D.

Lemma 2 is valid even if A is n-dimensional where A is replaced by A_i for $i = 1, \dots, n$.

We now have all the tools needed to establish the main proposition of this section.

Theorem 1: Given the distribution of voters' types, $g(x)$ and $h(c)$, such that $u \in C^1$, $H \in C^1$, their derivatives are bounded, $h(0) > 0$, and u is concave in A for all x and strictly concave for some x. If (A^*, B^*) is a strong rational election equilibrium, then $A^* = B^*$, $P_a = P_b = 1$, $Q_a = Q_b = 0$, and A^* maximizes $\int u(A, x)g(x)dx$.

PROOF: From Proposition 8 we know that if (A^*, B^*) is an equilibrium then so is (A^*, A^*) . We concentrate on the latter. Suppose that (A^*, A^*) is an equilibrium. We know that $\max W(A, A^*) \leq 0$ for all A. From Lemma 1 it must be true that $J = \int H((P_a/2)|D|)I(x)g(x)dx \leq 0$ for all A. From Lemma 2 and the first-order conditions for maximization it must be true, therefore, that

⁷For arbitrary functions $f(x)$, if $f'(x) \rightarrow a$ as $x \rightarrow 0$ for all sequences of x, then $f'(0) = a$. Let $A \rightarrow B$ so that $D \rightarrow 0$. Then $dH/dA \rightarrow h(0)(\frac{1}{2})(dU(A, x)/dx)$ for all such sequences.

$dJ/dA = 0$ at $A = A^*$. If $A = A^*$ then $D = 0$, $P_a = 1$, and

$$h(0) \int (du(A^*,x)/dA)g(x)dx = 0$$

Since $\int u(A,x)g(x)dx$ is a strictly concave function, A^* maximizes that function.

To finish the proof, we need to show that if (A^*,B^*) is an equilibrium then $A^* = B^*$. Suppose not. From Proposition 8, both (A^*,A^*) and (B^*,B^*) are equilibria. Therefore, both A^* and B^* are maximizers of $\int u(A,x)g(x)dx$. But u is strictly concave for some x which implies that there is a unique maximizer; that is, $A^* = B^*$.

Q.E.D.

Theorem 1 fully characterizes the rational-election equilibrium if it exists. In that equilibrium, even though no one votes -- thus avoiding all the nonproductive costs of voting -- candidates are led to select a platform which maximizes a social-welfare function, the sum of voters' utilities. The existence of voters who are on the margin of voting, those with costs near 0, leads candidates to take the preferences of these voters into account. Because of the linearity of utility in the costs of voting, the change in the probability that a voter will vote, due to a change in a candidate's position, is "locally" proportional to the extra utility received by the voter if that candidate is elected. It is always in the interest of the candidates to change their position in the direction which maximizes the "aggregate marginal utility of the marginal voters." This leads them inexorably to a position which maximizes the aggregate utility of the voters whose costs are minimal.⁸

Because of the similarity of this theorem to the fundamental-welfare theorem that competitive-market equilibrium allocations are Pareto-optimal, I am finding it difficult to refrain from phrases like

⁸If types and costs are correlated, that is if the density is $g(x,c)$ instead of $g(x)h(c)$, then candidates will choose the platform A^* which maximizes $\int u(a,x)g(x,0)dx$.

"the invisible hand of the electorate." However, the fact that equilibrium platforms maximize a "social utility function" should not lead the reader to conclude that election-equilibrium allocations are also Pareto-optimal. The next few examples help to illustrate this and other implications of the model.

Example 1: Suppose there is a one-dimension issue space and that the class of utility functions which any voter can have is given by $u(A,x) = -|A-x|$. x is usually interpreted to be voter x 's ideal platform. For this type of example, traditional theory tells us that the election equilibrium will be the ideal platform of the median voter, $A = x^*$ where $\int^{x^*} g(x)dx = 1/2$. Let us calculate the rational-election equilibrium. A^* will maximize $\int u(A,x)g(x)dx = \int -|A-x|g(x)dx$. It is easy to see that A^* will also be the median of the density $g(x)$. The two theories yield the same predicted-equilibrium platform, although turnout is predicted to be 100% by the traditional theory but 0% by this theory.

Example 2: Let us now look at a well-used example. Suppose that preferences over a one-dimension issue space are given by the Type-1 utility functions, $u(A,x) = -(A-x)^2$. In this case traditional theory still predicts that the platform will be the median voter's ideal platform. The rational-election equilibrium is, however, the mean voter's ideal platform. That is, A^* maximizes $\int -(A-x)^2 g(x)dx$. Differentiating, one gets $\int -2(A-x)g(x)dx = 0$. From this, we know that $\int Ag(x)dx = \int xg(x)dx$ or $A = \int xg(x)dx$, the mean of $g(x)$.

This simple example illustrates that there is absolutely nothing sacred about the median voter.⁹ One might just as easily be concerned about the mean or, indeed, any other moment. For example, if $u = -(A-x)^b$ then the $(b-1)$ st moment is the equilibrium platform. The predicted equilibrium platform depends on the composition of the class of utility functions. An important implication of this and the prior example is that functional forms are important. The functions $-|x-A|$ and $-(x-A)^2$

⁹Hinich (1977), Coughlin and Nitzan (1981), and Coughlin (1983a) also find the median to be unimportant when uncertainty is included in the voting model.

A)² each represent the same ordinal risk-free preferences on the set of A. However, they do represent different attitudes towards risk and different indifference surfaces between c and A. These differences are reflected in different equilibria. The intensity of preference for A as opposed to c, as measured by the willingness to vote, is what drives the result.

One other fact to note in this example: a multiple issue space will not eliminate this equilibrium. If A and x are, say, n-dimensional, then the equilibrium is the mean of the multivariate distribution g(x). These remarks reinforce the insights of Hinich (1978) that (a) in the presence of uncertainty functional forms are important and (b) quadratic-loss functions can imply that the mean ideal platform is the two-candidate equilibrium. Hinich bases his model on voters' errors in perceptions of candidates. My model shows that his conclusions hold even where errors are not present.

Example 3: Finally, let us look at a simple application of this theory and consider what happens if the election is held to decide the allocation of a public good and the assignment of the taxes needed to pay for that good. Let $u(y, I, x)$ be the utility of voter x for the public-good level, y, when that voter's income is I. We assume that x and I are independently distributed according to $r(x)$ and $s(I)$. Platforms will be of the form $(y, t(\cdot))$ where the function, $t(I)$, indicates the tax to be paid if income is I. I am assuming that taxes cannot be placed directly on the unobservable x. If the cost of the public good is $C(y)$, we require that $\iint t(I)r(x)s(I)dI dx = C(y)$ for all platforms -- no deficit or surplus financing is allowed. Given this model, we know that, in a rational-election equilibrium, y and $t(\cdot)$ maximize $\iint u(y, I - t(I), x)r(x)s(I)dI dx$ subject to the above constraint. Letting L be the Lagrangian multiplier associated with the constraint, it follows from first-order conditions that

$$d[\iint u(y, I - t(I), x)r(x)s(I)dI dx]/dy - L[dC(y)/dy] = 0,$$

$$- (I) \int (du(y, I - t(I), x)/dI)r(x)dx + L(I) \int r(x)dx = 0$$

for all I and

$$C(y) = \iint t(I)r(x)s(I)dI dx.$$

Let I^* solve $\int (du(y, I^*, x)/dI)r(x)dx = L \int r(x)dx$. The second equation above implies that in equilibrium $I - t(I) = I^*$ for all I; that is, $t(I) = I - I^*$. Everyone's after-tax income will be identical -- income is redistributed towards the mean. Using the constraint, we find that everyone's after-tax income is $I^* = (RM - C(y))/NR$ where $N = \int s(I)dI$, $R = \int r(x)dx$ and $M = \int Is(I)dI$. Turning to the first of the first-order conditions, it can be shown that if $d(du/dI)/dx = 0$, that is if du/dI is constant over all x, then

$$L = dU(y, I^*, x)/dI$$

and

$$N \int (du/dy)/(du/dI) r(x)dx = dC/dy.$$

This is simply the Samuelson-Lindahl first-order condition for the Pareto-optimal allocation of the public good: the sum of marginal rates of substitution equals marginal cost. Thus, we conclude that if the post-tax marginal utility of income is independent of the voter's type, then large two-candidate elections allocate resources efficiently. There are no "free riders" in this situation.¹⁰

Examples of utility functions for which $d(du/dI)/dx = 0$ can be given:

$$u = v(y, x) + I$$

$$u = v(y, x) + w(u, I), \text{ and, as a special case,}$$

$$u = x \ln y + \ln I.$$

I leave it to the interested reader to show that if income and type are not independent, then the efficiency disappears and redistribution will no longer require equal post-tax income. One can also show that if costs of voting and income are positively correlated, as is sometimes argued, then low-income types will have a larger impact on redistri-

¹⁰A side issue: since this case covers utility functions without income effects, it covers all situations covered by the Demand-Revealing Mechanisms. Therefore, it dominates that method for social choice.

bution.

These three examples are only a small indication of the powerful use one can make of the rational-election equilibrium. I am sure the eager reader can provide many more.

To prove that all the above is not vacuous, we move next to the question of existence.

V. EQUILIBRIUM AND EXISTENCE

In the traditional theory of majority-rule equilibrium with no abstentions, existence of equilibrium is an unusual occurrence. One implication is that we cannot rely on theorems which assume existence. For example, local public-goods theories using the median voter should be highly suspect; the results are likely to be vacuous. The equilibrium described in this paper, on the other hand, exists in many cases. These equilibria can potentially provide the foundation for many models which make social choices by majority-rule elections.

In the last section we proved that if A^* were a rational-election equilibrium, then A^* maximized aggregate utility. If we could prove the converse, that if A^* maximizes aggregate utility then A^* is a rational election equilibrium, we would be done since the appropriate compactness and continuity conditions which ensure the existence of a maximum (the Weierstrauss Theorem) are well-known. Unfortunately, the converse is not true without additional conditions on the densities g and h .¹¹ It is our task to delineate as much as possible the set of distributions for which the following is true:

(S) if A^* solves $\max \int u(A,x)g(x)dx$, then A^* is a rational-election equilibrium.

If we knew for which (g,h) the function $W(A,B)$ were concave in A and convex in B , with $V(A,B)$ behaving symmetrically, we would be done, since under these conditions the game-theoretic solution to the

¹¹Economists will notice that this phenomenon also arises when considering the welfare theorems about competitive equilibria.

candidates' problem is known to exist. Unfortunately, one cannot take this approach. Remember that $W(A,B) = Q_a - Q_b$ where $Q_a = \int_{X^+} H((P_a/2)(D))g(x)dx$ and $Q_b = \int_{X^-} H((P_b)(-D))g(x)dx$ and where D is concave in A and convex in B from the concavity of u (leaving aside the behavior of P_a and P_b for the moment). If H is a concave function of c then Q_a is concave, but we cannot tell about Q_b which, in this instance, is a concave function of a convex function. If H is convex, then we have a symmetric problem since $-Q_b$ is concave, but we can say nothing about Q_a . Only if H is linear, both concave and convex, can we discuss the concavity properties of W . We capture this intuition in the next proposition.

PROPOSITION 9: If $h(\cdot)$ is the uniform density on $[0,k]$, $k > 0$, then (S) is true.

PROOF: Let $J = \int H(P_a/2) |D| I(D)g(x)dx = \int (1/k)(P_a/2) |D| I(D)g(x)dx = \int (1/k)(P_a/2) DG(x)dx$. At A^* , the maximizer, letting $D = u(A,x) - u(A^*,x)$, we see that $J = 0$. At any other A , $J \leq 0$. Referring to lemma 1 we now conclude that A^* is a rational-election equilibrium.

Q.E.D.

Absolutely no conditions have been placed on g . That is, we need not worry about single-peakedness, symmetry, unimodality, or unidimensionality. Any density over concave-utility functions can be accommodated. The second thing to notice is that we have been precise about h . An obvious question is whether (S) is true when h is not uniform. The answer is "no" if we require all g to be accommodated.

PROPOSITION 10: (S) is true for all g if and only if h is uniform.

PROOF: The "if" is simply Proposition 9. We prove the "only if" statement.

Suppose we have a nonuniform h , an A , a g and an A^* such that A^* solves $\max \int u(A,x)g(x)dx$ and such that $\int H((P_a/2)|D|)I(D)g(x)dx \leq 0$ when $D = u(A,x) - u(A^*,x)$. If there are no such g , h , and A then we are done, since there will then be no g for which (S) is true. If there are such a g , h , and A for which (S) is true, we can show that there will be another g for which (S) is not true, which would prove the proposition. Thus, if we show that we can perturb g to g' such that $\int g'Ddx \leq 0$ and $\int H((P_a/2)|D|)I(D)g'dx > 0$, then we will have proven that (S) is

false and the proposition is true.

The perturbation works as follows. Let S_1 be the set of x for which $D > 0$ and $H(c) > c/k$, let S_2 be those x for which $D > 0$ and $H(c) < c/k$, let S_3 be the x for which $D < 0$ and $H(c) > c/k$, and let S_4 be the set of x such that $D < 0$ and $h(c) < c/k$. We make g larger on S_1 and S_3 and smaller on S_2 and S_4 by letting $g'(x) = g(x) + e_i$ when $x \in S_i$, such that $\int e_1 dx + \int e_2 dx + \int e_3 dx + \int e_4 dx = 0$, such that $e_1 \int D dx = e_2 \int D dx = e_3 \int D dx + e_4 \int D dx \leq 0$, and such that $e_1 \int H((P_a/2)D)I(D) dx + e_2 \int H dx + e_3 \int H dx + e_4 \int H dx > 0$. The careful reader can check to see that as long as H is not uniform, this perturbation will be possible, since the sets S_i will be nonempty.

Q.E.D.

Proposition 10 informs us that if we want a simple-existence theorem and we want it to be applicable to all possible preference patterns, we must restrict our attention to uniform distributions of costs. Suppose, instead, we want a theorem applicable to all distributions of costs. The answer is similar to that in Proposition 10 -- statement (S) is true for all cost distributions if and only if we severely restrict the possible preference distribution g . In order to see why, let us first define the derived distribution of utility differentials. Let $J(r) = \int \chi(r) g(x) dx$ where $\chi(r) = [x | u(A,x) - u(A^*,x) \leq r]$, and let $j(r) dr = dJ(r)$. Finally let $l(r) = j(r) - j(-r)$ for all $r \geq 0$. It can be shown that A^* maximizes $\int u(A,x)g(x) dx$ if and only if

$$\int_0^{\infty} r l(r) dr \leq 0 \text{ for all } A. \quad (5.1)$$

It can also be shown that expected plurality $W(A, A^*) =$

$$\int_0^{\infty} \{H((P_a/2)r)/r\} r l(r) dr. \quad (5.2)$$

Statement (S) is true when (5.1) implies that (5.2) is less than or equal to 0. Therefore, in order for statement (S) to be true, the function $H((P_a/2)r)/r$ cannot weight r relatively more heavily when $l(r) > 0$ than when $l(r) < 0$. Notice that an equal relative weighting occurs exactly when H is uniform. If we require that (S) be true for all possible h , then we must not allow $l(r) > 0$, for otherwise there will be

at least one H which weights $l(r)$ incorrectly. We capture all of this in the following:

PROPOSITION 11: (S) is true for all h if and only if $\int_2 l(r) dr \leq 0$ for all $z \geq 0$, and all A at A^* .

PROOF: (IF) Let (5.1) be true and let $z_1 = \sup \{r | l(r) > 0\}$. If $l(r) \leq 0$ for all r , then we are done since $\int (H/r) r l(r) dr \leq 0$. Therefore, we consider the case for which $l(r) > 0$ for some $r < \infty$. If $z_1 = \infty$, then there is a z' such that $\int_2 l(r) dr > 0$. But this is impossible by hypothesis. Thus, we need only consider cases for which $z_1 < \infty$.

Let $z_2 = \sup \{r | 0 \leq r \leq z_1, l(r) > 0\}$. Let $I_1 = [a_1, \infty]$ and $I_2 = (z_2, z_1)$. Since $\int_2 l(r) dr \leq 0$, it follows that $\int_{I_2} l(r) dr \leq 0$. Further, since H is a distribution function, $H(kz) \geq H(kz_1)$ if $z \geq z_1$ and $H(kz) \leq H(kz_1)$ if $z \leq z_1$. Therefore, letting $p = P_a/2$,

$$\int H(pr) l(r) dr = \int_{-\infty}^{z_1} H(pr) l(r) dr + \int_{z_1}^{\infty} H(pr) l(r) dr \leq H(p(z_1)) \int l(r) dr \leq 0.$$

One can iterate this proof for all z_1 until $z_1 = 0$.

(only if) Suppose that $\int_2 l(r) dr > 0$ for some $z^* \geq 0$. Let $H''(z) = 1$ if $z \geq z^*$ and $= 0$ if $z < z^*$. Then $\int \{H''((P_a/2)r)/r\} l(r) dr = \int l(r) dr > 0$. But then A^* cannot maximize $\int u(A,x)g(x) dx$.

Q.E.D.

In the proof of Proposition 11, we used a distribution of costs, H'' which is very discontinuous at z^* . We could, however, have found a continuous H'' which is near to H'' and which is also appropriate. Thus, the above proof would remain applicable with minor adjustments.¹²

In this section, we have proven results only about the extreme limits of the set of (g, h) for which rational-election equilibrium

¹²As a side note, the cost distribution, H'' , used in this proof, which assumes equal costs of voting known to all, is the same distribution used in Ledyard (1981). The fact that this distribution causes the most difficulties for existence partly explains the weak theorem in that paper.

exists. That is, we have required existence to occur either for all g or for all h . If we are willing to consider only some g or h , we should do better. One can show that there is an open set of (g,h) for which existence obtains. In particular, if h is almost uniform or if g is almost "symmetric," then equilibrium will exist. I suspect that there is a large set of such (g,h) , but its precise characterization remains an open question.

Since the results of Coughlin; of Hinich, Ledyard, and Ordeshook; of Hinich; and of this paper all point to the fact that multidimensional election equilibria exist more often than suspected and that they rarely involve the median voter, one might speculate whether it is my assumption of rationality or the role of uncertainty which derives these results.¹³ I suspect uncertainty is the key to existence and that some form of rationality is the key to "optimality." This remains a future research issue.

VI. VARIATIONS ON A THEME

As I have presented this paper in many places, a number of issues have been raised which seem to be easily handled within the framework of the above model. Let me address these variations.

(1) Income effects

In the analysis of the rational voter I assumed that the cost of voting entered the voter's utility function linearly. This assumption is not necessary and can be eliminated. In particular, let $u(A,0,x)$ be the utility received by the voter if A wins and this voter did not vote. Let $u(A,c,x)$ be the utility if A wins and this voter voted where x and c are as in the original model. Assume that du/dc exists and is less than zero (that is, an increase in the cost of voting lowers x 's utility, ceteris paribus). Although the analysis is messier than above, one can derive similar results. For example, at an equilibrium

¹³These remarks are motivated by an insightful observation of Howard Rosenthal.

$$h(0) \int u_A \{ (A,0,x) / -u_c(A,0,x) \} g(x) dx = 0,$$

This is identical to the earlier result if we "normalize" marginal utility by the marginal utility of voting costs at 0. That is, if an equilibrium exists $A^* = B^*$ and A^* maximizes

$$\int [u(A,0,x) / u_A(A^*,0,x)] g(x) dx.$$

I do not yet know how other results translate. For example, establishing existence appears to be more difficult.

(2) Negative voting costs

I am suspicious of anyone who claims to vote no matter what the issues or how close the election. In almost every election there are frictions, or other phenomena ignored by this model, which cause differences among the candidates and which might lead low-cost voters to vote. As far as I can tell there is still no common agreement on the facts about voter behavior. In spite of my skepticism it is important for completeness of the theory to explore what would happen to the equilibrium if there were indeed voters who derive utility from the act of voting itself. It is easiest to model these as voters whose cost, c , is negative. In the model of the calculus of voting, a voter with $c < 0$ will always vote for his most preferred candidate. With this in mind consider now the equilibrium in which those voters with $c < 0$ always vote and those voters with $c > 0$ behave as described above. If $A = B$ then only the voters with $c < 0$ will vote and, therefore, if $A = B$ in equilibrium it must be true that A is the ideal platform of the median voter, the median of those who always vote, if one exists. (We know from standard theory that existence can be problematical.) If c and x are uncorrelated and if that median platform also maximizes $\int u(A,x)g(x)dx$, then A will be the equilibrium. However, if the median either does not exist or does not equal the maximizer of aggregate utility, then we must look elsewhere. It is an open question as to whether an equilibrium even exists in this situation and, if so, what it is. I am not even sure whether candidates would choose the same platform in equilibrium. All I can conclude so far is that "irrational" voters who derive utility from the act of voting create an externality

which interferes with the selection by the election of a socially desirable outcome. Perhaps we should educate voters not to be "citizens" but to be selfish?

(3) Minimax-regret voters

Suppose some voters in the electorate use the minimax regret criteria of Savage (made popular by Ferejohn and Fiorina 1974, 1975). The analysis remains much the same, but the conclusions are slightly altered. As I showed in Ledyard (1981), if we replace P_a by $1/2$ in the model of rational-voting behavior we will have modeled the behavior of a minimax-regret voter. If one then follows the model to its conclusion, one will see that, in equilibrium $A^* = B^*$, no one will turn out, and A^* will maximize $\int u(A,x)[g(x) + (1/2)g^*(x)]dx$ where $g(\)$ is the density of the expected utility-maximizing voters and $g^*(\)$ is the density of the minimax-regret voters. It appears that because minimax voters do not care about closeness, they end up being weighted at half that of utility maximizers in their effect on the outcome. At the margin, when A is near B , they react more slowly to changes in platforms and, thus, lose their effectiveness.

(4) Vote-maximizing candidates

It is sometimes argued that candidates care about other things than just winning. This is another of those areas of disagreement in political theory. There is no agreement on the factors which motivate candidates. Although it is obviously of little use to a candidate to have a large vote if that candidate does not win, some argue that candidates should want to maximize votes, not the probability of winning. Several of our conclusions change if that is the case. First, candidates will not choose the same platform. If they did, one of them could increase his votes (from 0) by simply moving away from the other candidate. (Of course, this could lead to an election loss.) In equilibrium, if one exists, turnout will occur with vote-maximizing candidates.

(5) Turnout

A major issue raised by many who see this model for the first time is the lack of turnout in equilibrium. While I see this as good (the deadweight loss of voting costs is avoided), many see this as a pre-

diction of the model clearly contradicted by the facts. It must be remembered that, because of the many possible frictions, actual elections will rarely match this theory. Among other things, most elections are held to decide several contests simultaneously and political activists, ignored in my model, operate to interfere with the natural forces. It is true that single-issue elections with few activists and with low stakes have little turnout. Examples abound but the normal school-tax election is the obvious one. In a New Hampshire town an election was held to fill the school board. Only one slate was on the ballot. No one voted. I am not sure why the judges did not write in their own names, but the moral is clear; when there is no choice it pays not to go to the polls.¹⁴

I am not sure what an appropriate example is for the model in this paper, but the following provides ease of computation. Let $u = -(A-x)^2$, $H(\) = 1 - \exp(-ac)$, and let $g(\) = (R)(b) \exp(-bx)$. Going through the appropriate manipulations one can, somewhat tediously, discover that given the platforms A and B , the maximum turnout $M(\)$ is

$$1 + (b/(aD-b))((\exp(-2aDS) - ((2aD/(aD+b))\exp(-bS))).$$

Here, $D = (A-B)/2$ and $S = (A+B)/2$.

It can be easily shown that M is near 1 if D, S, a, b are large. M is near 0 if D, S, a, b are near 0. I have no idea what "reasonable" values of these parameters are. Does anyone want to guess?

If one wishes to estimate equilibrium turnout, given A and B , one must solve the following equation; let $M(a, b, D, S)$ by the equation above, then solve $N = M(a/N, b, D, S)$ for N . N/R will then be an estimate of the percentage turnout since $1/N$ estimates P_a .

These are but a few of the possibilities for refinement of the model. Others follow which I think are as important, but of which I know little:

¹⁴Stop Press: For those who believe the probability of a tied election is empirically zero, let me report the outcome of the 1983 election for the Board of Trustees of Pasadena Community College: Gertmenia, 2592 and Miele, 2592. This followed a recount. Both candidates argued against drawing lots, the legislated action, as being undemocratic.

- (a) Three-candidate elections (and multiple-candidate)
- (b) Political activists and parties
- (c) Candidate choice and the role of primaries
- (d) Intertemporal considerations
- (e) Representative democracy and the responsiveness of the system
- (f) Multiple, simultaneous elections
- (g) Empirical estimation

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